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TWO ELEMENTARY CONSTRUCTIONS IN COMPLEX TRIGONOMETRY

By A. E. Kennelly

It is well known, and is easily shown, that the coordinates x and y of any point of the rectangular hyperbola $x^2 - y^2 = 1$ are equal respectively to $\cosh(2A)$ and $\sinh(2A)$, where A is numerically equal to the area bounded by the axis of X, the hyperbola, and the radius-vector of the point (x,y); while the coordinates x and y of any point of the circle $x^2 + y^2 = 1$ are equal respectively to $\cos(2A)$ and $\sin(2A)$, where A is numerically equal to the area bounded by the axis of X, the circle, and the radius-vector of the point (x,y).

In figures 1 and 2, the shaded areas are each numerically equal to one half. In fig. 1, $Ob = \sin 1$ and $bB = \cos 1$. In fig. 2, $Ob = \sinh 1$ and $bB = \cosh 1$.

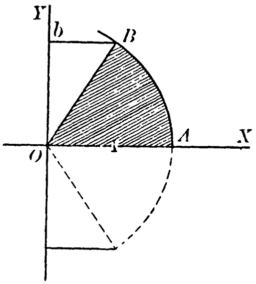

Fig. 1

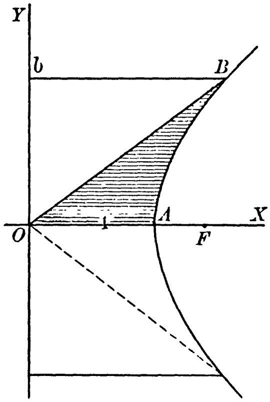

Fig. 2.

Assuming the familiar relations

$$\sin j\alpha = j \sinh \alpha,$$

$$\cos j\alpha = \cosh \alpha,$$

where $j = \sqrt{-1}$, we have

$$\sin (\alpha + j\beta) = \sin \alpha \cosh \beta + j \cos \alpha \sinh \beta,$$

and

$$\cos (\alpha + j\beta) = \cos \alpha \cosh \beta - j \sin \alpha \sinh \beta;$$

hence, $\sin(\alpha + j\beta)$, in the complex plane, is a point of intersection of the ellipse

$$\frac{x^2}{\cosh^2\beta} + \frac{y^2}{\sinh^2\beta} = 1 \qquad (1)$$

and the hyperbola

$$\frac{x^2}{\sin^2\alpha} - \frac{y^2}{\cos^2\alpha} = 1, \qquad (2)$$

while $\cos(\alpha + j\beta)$ is a point of intersection of the ellipse (1) and the hyperbola

$$\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1\,; \qquad (3)$$

the foci of all these curves being the points $(\pm 1, 0)$.

These facts suggest the following constructions:

1. Construction for $\sin(\alpha + j\beta)$. (Figure 3). Take $OA = 1$ along the negative end of the Y-axis. From OA as initial line, mark off the circular area $AOB = \alpha/2$. From OB as initial line, mark off the hyperbolic area $BOD = \beta/2$ Let C be the foot of the perpendicular from D on OB produced. Drop perpendiculars from C and D on the axis of reals OX, at c and d respectively. About c as centre, rotate cd positively through 90° to cZ. Then will the complex vector $OZ = Oc + jcd$ be the required sine of the complex angle $\alpha + j\beta$.

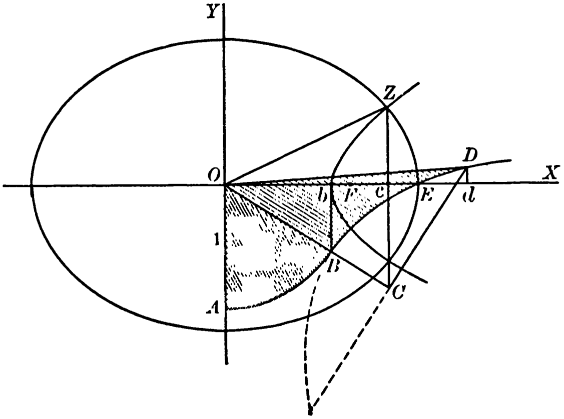

FIG. 3.

In the case represented, $\sin(1 + j1) = 1.299 + j0.635 = 1.446 \operatorname{cis} 26^\circ\, 3'$.

As β varies, Z moves along the hyperbola (2); as α varies, Z moves along the ellipse (1).

From the figure we get at once (if $x = Oc$ and $jy = cZ$),

$$\begin{aligned}\sin^{-1} OZ = \sin^{-1}(x \pm jy) &= \sin^{-1} Ob \pm j \cosh^{-1} OE \\ &= \sin^{-1}\left\{\frac{\sqrt{(1+x)^2+y^2} - \sqrt{(1-x)^2+y^2}}{2}\right\} \\ &\qquad \pm j\cosh^{-1}\left\{\frac{\sqrt{(1+x)^2+y^2} + \sqrt{(1-x)^2+y^2}}{2}\right\},\end{aligned}$$

since Ob is the semi-transverse axis of the hyperbola (2) through Z, while $OE = OC$ is the semi-major axis of the ellipse (1) through Z.

2. Construction for $\cos(\alpha + j\beta)$. (Figure 4.) Take $OA = 1$ along the positive end of the axis of reals. From OA as initial line mark off the circular area $AOB = \alpha/2$, etc., precisely as in the preceding paragraph. The complex vector $OZ = Oc + jcd$ thus obtained will be the required cosine of the complex angle $\alpha + j\beta$.

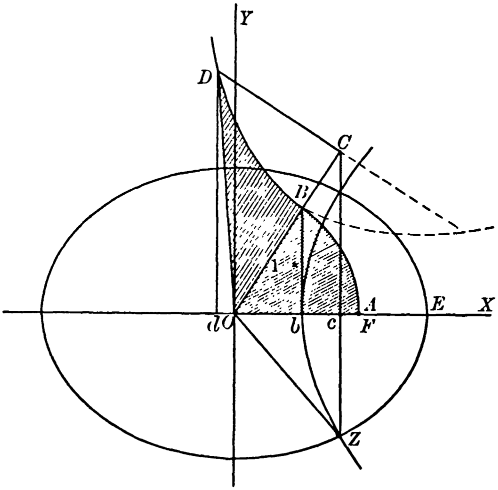

FIG. 4.

In the case represented,

$$\cos(1 + j1) = 0.834 - j\,0.989 = 1.293 \operatorname{cis} 49^\circ 52'.$$

As β varies, Z moves along the hyperbola (3); as α varies, Z moves along the ellipse (1).

From the figure,

$$\cos^{-1} OZ = \cos^{-1}(x \pm jy) = \cos^{-1} Ob \mp \cosh^{-1} OE$$

$$= \cos^{-1}\left\{\frac{\sqrt{(1+x)^2+y^2}-\sqrt{(1-x)^2+y^2}}{2}\right\}$$

$$\mp j\cosh^{-1}\left\{\frac{\sqrt{(1+x)^2+y^2}+\sqrt{(1-x)^2+y^2}}{2}\right\}.$$

It is evident that the same methods of construction serve also for $\sinh(\alpha + j\beta)$ and $\cosh(\alpha + j\beta)$. For, from $\sinh \theta = -j\sin(j\theta)$ and $\cosh\theta = \cos(j\theta)$,

we have, $$\sinh(\alpha + j\beta) = j\sin(\beta - j\alpha),$$

and $$\cosh(\alpha + j\beta) = \cos(\beta - j\alpha).$$

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